Developments in Research on Mathematical Practice and Cognition

Alison Pease, a Markus Guhe, b Alan Smaill b

a Imperial College London
b School of Informatics, University of Edinburgh

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Abstract
We describe recent developments in research on mathematical practice and cognition and outline the nine contributions in this special issue of topiCS. We divide these contributions into those that address (a) mathematical reasoning: patterns, levels, and evaluation; (b) mathematical concepts: evolution and meaning; and (c) the number concept: representation and processing.

Keywords: Mathematical practice and cognition; Mathematical reasoning; Mathematical concepts; The number concept; Cognitive science of mathematics; Philosophy of mathematical practice

1. Introduction
This topiCS issue grew out of a Symposium on Mathematical Practice and Cognition at the AISB congress held in Leicester in 2010, in which researchers from different disciplines with an interest in mathematical practice and cognition came together with the goal of sharing ideas and finding interdisciplinary connections. Thus, for the most part, these papers differ from traditional work in the cognitive science of mathematics in that they offer insights from the philosophy, history, and sociology of mathematics, in which the activity of mathematics is interpreted to include informal research-level mathematics. The main aim of this issue is to present these new developments in research on mathematical practice and cognition to a wider cognitive science audience.

Correspondence should be sent to Alison Pease, Imperial College London, 180 Queens Gate, London SW7 2RH, United Kingdom. E-mail: A.Pease@imperial.ac.uk
The study of mathematical practice and cognition is a multi-disciplined endeavor with associated multiple answers to the questions of what the objects of study are, which methodologies are appropriate for studying them, and what the purpose of the endeavor may be. Cognitive science of mathematics has a short but productive tradition, with advances in mathematical idea analysis and numerical cognition. Although the focus is usually on early, basic or developmental mathematics, this has occasionally been extended, most notably by Lakoff and Núñez, who use techniques from linguistics to analyze language in advanced mathematics, (2000). Much has been achieved in the study of mathematical thinking by crossing disciplinary boundaries. Criticizing the dominance of formalism in the philosophy of mathematics, Lakatos famously paraphrased Kant: “the history of mathematics, lacking the guidance of philosophy, has become blind, while the philosophy of mathematics, turning its back on the most intriguing phenomena in the history of mathematics, has become empty” (Lakatos, 1976). This has led to the development of a new field, which lies at the junction of the history and philosophy of mathematics: the philosophy of mathematical practice. Subsequent researchers within this new field have extended Lakatos’s plea by looking at current, as well as historical, mathematical practice, and studying ways in which research mathematicians reason and communicate, including work on visualization (Giaquinto, 2007; Mancosu, Jørgensen, & Pedersen, 2005), analogies (Bartha, 2010; Schlimm, 2008), and concept development (Tappenden, 2008, 2008)—topics that have traditionally been in the realm of cognitive science. These studies of mathematical practices typically focus on informal mathematics, as opposed to polished presentations, and have led to a slew of publications with a strong interdisciplinary flavor, such as Tymoczko’s (1998) New directions in the philosophy of mathematics (Tymoczko, 1998), which includes essays from philosophers, mathematicians, logicians, and computer scientists; publications deriving from two Perspectives on Mathematical Practices conferences (van Kerkhove & van Bendegem, 2004, 2007; van Kerkhove, 2009; van Kerkhove, de Vuyyst, & van Bendegem, 2010), which relate the philosophy of mathematics to the history of mathematics, and propose connections between a philosophy of mathematical practice and the psychology and sociology of mathematics; and Philosophy of Mathematics: Sociological Aspects and Mathematical Practice (Löwe and Müller, 2010), which records the progress of an interdisciplinary network of researchers aiming to combine tools from sociology, psychology, educational studies, and history to provide a philosophy of mathematical practice.

Our focus on interdisciplinarity in this issue has manifested itself in various ways. Many of the papers make interdisciplinary arguments; for each submission we allocated reviewers from different disciplines, to bring new perspectives to the work; and the collection includes contributions from cognitive scientists, psychologists, philosophers and historians of mathematics, researchers in mathematics education, and researchers in artificial intelligence. Of course, even though this issue is strongly interdisciplinary and all articles make a substantial effort to cross the boundaries of scientific disciplines, each article has to be understood with respect to the discipline from which it originates.
The articles in this issue can be arranged along many dimensions, for instance:

- Empirical versus theoretical research;
- Mental processing of numbers (numerical distance) versus the cognition of mathematical concepts;
- Mathematical education versus research mathematics.

The *empirical—theoretical* dimension addresses different views on mathematics: from an empirical viewpoint, mathematics is a particular kind of human behavior that can be investigated by observation and statistical modeling. From a theoretical viewpoint, mathematics raises issues such as the epistemological status of theorems and other mathematical objects.

The *number—concept* dimension is mainly one of granularity. In particular, research in psychology and research on the brain has focused on issues of how numbers (comparatively concrete mathematical concepts) are processed, most notably what cognitive/brain processes are involved in comparing numbers of different magnitudes. Such a problem lends itself well to the typical methodologies of experimentation and statistical analysis used in psychology and the neural sciences. Applying these methods is more difficult for high-level concepts of mathematics; therefore, research on such issues tends to be carried out in more theoretical disciplines like philosophy or informatics (often in the form of automated reasoning).

The *education—research* dimension spans research on mathematical education versus what we call *research mathematics*. Mathematics is an acquired skill and there are many issues involved in understanding how this process works, in particular when it comes to problems of how best to teach mathematics. However, the reverse is also true: Understanding how people acquire mathematics enhances our understanding of cognition in general. Apart from transmitting mathematical ideas from one head to another, however, there is also the issue of how mathematics came about in the first place, and this seems to be a creative process carried out by mathematicians.

This issue contains nine contributions and, for the purposes of this introduction, we use a rough trifold division of articles into ones that address:

1. Mathematical reasoning: patterns, levels, and evaluation;
2. Mathematical concepts: evolution and meaning;
3. The number concept: representation and processing.

### 2. Mathematical reasoning: Patterns, levels, and evaluation

Aberdein argues, in “Mathematical Wit and Mathematical Cognition,” that humor in mathematics can be used to provide an insight into cognitive processes and can be the reward for successfully navigating certain defeasible cognitive heuristics. He takes a piece of mathematical folk humor constituting a list of spurious proof types which is well known in mathematics departments, and argues that the list has some grounding in reality
by showing some historical examples of mathematical practice in which similar proofs have been suggested. Aberdein suggests that one role which the jokes play is the acculturation of student mathematicians, supporting his argument with findings from social psychology that show that humor can be used as a way of socializing members into accepting norms which are important to a group. The main contribution of his work lies in his pairing of the jokes to argumentation schemes, or patterns of informal reasoning, by seeing each spurious proof in terms of one or more schemes and offering critical questions in each case. Argumentation theory is not usually developed with the goal of describing mathematical reasoning, although there has been some work on applying the schemes to mathematics, so Aberdein’s use of the jokes to help to identify schemes which are commonly used in mathematical inference, and the critical questions to highlight flaws in the reasoning, is particularly interesting. He concludes with some possible consequences for AI, in terms of simulating creativity in mathematics.

Schiller’s paper “Granularity Analysis for Mathematical Proofs” addresses the issue of granularity in mathematical proof: A proof can be conveyed at different levels of detail, from an outline sketch to a formal, fully annotated proof where every step is justified at the level of primitive inference rules. Mathematicians hardly ever produce such full formal proofs, and in a pedagogical context they will use more or less detail depending on the assumed knowledge and reasoning ability of the student. A system that can combine the rigor of machine proof with the ability to make sense of students’ attempted proofs at varying degrees of granularity, and give feedback at the appropriate level, would clearly be a valuable tutoring resource. Schiller’s paper describes an approach to this problem and outlines how this approach may benefit the computer-supported teaching of proofs in the context of university-level education.

Inglis, Mejia-Ramos, Weber, and Alcock use empirical methods in “On Mathematicians’ Different Standards When Evaluating Elementary Proofs” to test the philosophical debate about whether mathematicians agree about the validity or invalidity of a proof, challenging some assumptions made by researchers in mathematics education that there is agreement among mathematicians. They asked 109 mathematicians whether a particular proof of undergraduate-level calculus was valid, how certain they were of their answer, and whether they would change their mind, given a reason for which it should be considered invalid. As hinted in the title of their article, their findings suggested, among other conclusions, that there was disagreement among mathematicians about proof validity. Inglis et al. then consider what effects differing ideas by educators about what constitutes a valid proof might have on mathematics students, and they spell out some consequences for the teaching of mathematics.

3. Mathematical concepts: Evolution and meaning

In “Mathematical Practice and Conceptual Metaphors: On Cognitive Studies of Historical Developments in Mathematics,” Schlimm, a philosopher and historian of mathematics, takes a critical look at arguments made in cognitive science and in particular the source
material that is commonly used. He alerts us to the dangers of forming theories based on material in textbooks or popular books about mathematics, in which a polished rational reconstruction of research mathematics is offered, and presenting them as theories about the messy informal processes which underlie research mathematics: that is, confusing Hersh’s “front” and “back” of mathematics (described in Hersh, 1991). Although clearly dependent on each other, Schlimm highlights gaps between the front and the back of mathematics, showing that conclusions which are formed from an analysis of the front cannot be assumed to hold for the back. He demonstrates his argument with a short history of the concept “set” and contrasts it with the mathematical idea analysis of the concept presented by Lakoff and Núñez.

In “The Motion Behind the Symbols: A Vital Role for Dynamism in the Conceptualization of Limits and Continuity in Expert Mathematics,” cognitive scientists Marghetis and Núñez take the unusual position of using history of mathematics to inform their cognitive theories—in particular, they analyze language used by Cauchy in the 19th century. This constitutes a truly inter-disciplinary perspective and contrasts the often narrow focus in cognitive science on mathematics as arithmetic and number representation. They complement their work with analysis of co-speech gestures used by contemporary graduate mathematics students while engaged in proving a theorem, on the assumption that both these and Cauchy’s language help to indicate underlying conceptual systems used during inference. They point out that such conceptual systems may be implicit and may be inconsistent with explicit, rigorous definitions. By examining reasoning about mathematical continuity, Marghetis and Núñez argue that while the standard $\varepsilon - \delta$ characterization suggests a static conceptualization, underlying dynamic conceptualization is sometimes in evidence. The interplay between such conceptual systems and rigorous definitions is an important aspect of mathematical practice, and thus an essential part of understanding the human cognitive activity of mathematical thought.

4. The number concept: Representation and processing

In “A Computational Modeling Approach on Three-Digit Number Processing,” Huber, Moeller, Nuerk, and Willmes describe a computational cognitive model of three-digit number processing. In this paper, they expand an earlier computational model of two-digit number comparisons to include the case of three-digit number comparisons. The model is a three-layer connectionist network. Moreover, they evaluated how different compatibility effects influence performance. (The compatibility effect is the finding that number comparisons are faster if, for example, the decade and unit digits of two-digit numbers differ in the same way, for example, comparing 45 and 78 is faster than comparing 54 and 78.)

Moeller, Klein, and Nuerk present evidence in their paper—“Influences of Cognitive Control on Numerical Distance Effects in Two-Digit Number Magnitude Comparison”—that number magnitude comparison may be more complex and not as automatic as it is generally assumed to be. They discuss an experiment that suggests that cognitive control exerted a strong influence on number magnitude comparison—probably the most widely
investigated and one of the most robust effects found in numerical cognition. They explain their finding that the frequency of specific unit digit combinations has an effect on how these numbers are processed in terms of an associative learning account of cognitive control.

The paper by Fisher, Riello, Giordano, and Rusconi on “Singing Numbers... in Cognitive Space—A Dual-Task Study of the Link Between Pitch, Space, and Numbers” explores the relationship between the domains of spatial—numerical and spatial—musical correlations, where the former is represented by the so-called SNARC effect, and the latter refers to an association between musical pitch and vertical alignment in space. Besides reporting experimental evidence that suggests that in neither case is automaticity involved, the paper gives a short introductory survey of work on the spatial—musical effect. The results are discussed in view of the differences as well as similarities between the domains, and the possible wider implications.

Mandelbaum argues, in “Numerical Architecture,” for a modular “number sense.” By reviewing evidence from different disciplines of cognitive science and using standard criteria for cognitive modularity (domain specificity, informational encapsulation, neural localizability, being subject to specific pathological breakdowns, mandatoriness, speed, and being inaccessible at the person level), he examines behavioral, neuropsychological, philosophical, and anthropological findings to argue for the existence of a number module. In particular, he makes the case for the accumulator processing model as a model for the number module.

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References


